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SPECTRAL RESPONSE OF AN ELASTIC SPHERE TO DIPOLAR POINT-SOURCES

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ABSTRACT

A stratified elastic sphere is excited by an harmonic dipolar source of arbitrary orientation and depth. The total field is expanded in series of vector spherical harmonics and then condensed into a convenient form of a displacement dyadic. The Haskell-Gilbert matrix method is employed to obtain the radial factor of the displacements for a multilayered sphere. The dependence of the field on the azimuth angle and the fault elements is obtained for the case of a double-couple at depth. Expressions are also developed for the radiation pattern of surface waves over a spherical stratified earth.

Introduction

The numerical solution of Lamb's problem for a stratified sphere will undoubtedly equip seismologists with a useful working model. Most studies in this direction were hitherto restricted to calculations of periods and dispersion curves. In some cases, relative spectral amplitudes were computed for very special sources such as torques, volume injections and similar devices which were chosen solely to excite a particular part of the total field and simplify the analysis.

Recent studies of earthquake sources from spectrums of surface waves disclosed that the equivalent force system as seen through the spectral window 50–400 sec at the far field is that of a double-couple. Analysis of first motions of body waves from numerous shocks also favoured the dipolar source, although not necessarily of the double-couple type.

To meet the needs of future observations, a combination of realistic sources and earth models is required. As a first step toward this goal, Ben-Menahem and Harkrider (1964) gave numerical solutions for the spectral response of a stratified half-space to dipolar excitations. The present paper consists of a theoretical extension of the former results to ultra-long surface waves ($T < 500$ sec) which are affected by the earth curvature but do not experience the effects of the core and the gravitational forces.

Propagation of seismic waves from point sources in a sphere were considered from various angles. Jobert (1955) investigated the curvature effect on Love waves in a spherical layer. Yanovskaya (1958) followed with the curvature effect on Rayleigh waves excited by a vertical singlet. Pekeris and Jarosch (1958) computed spheroidal free oscillations for a real earth model. Gilbert and MacDonald (1960) extended the Haskell matrix method to a layered sphere and computed toroidal oscillations and surface amplitudes for various earth models.

In this paper we have adapted with minor changes the nomenclature of Gilbert and MacDonald (1960). A spherical system is set up at the center of a sphere of radius $R = a$. Three unit vectors $(\mathbf{a}_R\mathbf{a}_\theta\mathbf{a}_\varphi)$ are drawn in the direction of increasing R, θ and φ so as to constitute a lefthand base system.

The basic solutions of the scalar wave equation $\nabla^2\psi + K^2\psi = 0$ are $z_l(KR)$ $P_l^m(\cos\theta)e^{\pm im\varphi}$ where in our case $z_l(KR)$ refers to the spherical Bessel function of the second kind and $P_l^m(\cos\theta)$ the associated Legendre functions of the first kind, order l and degree m (defined as in Morse and Feshbach, 1953). The symbol ${}_n\omega_l$ designates the angular frequency of the n-th normal mode of order l.

Basic Solutions and Source Representation

The vibrations of an elastic sphere may be expressed as a linear combination of three vector functions **L**, **M**, **N**

$$\mathbf{L}_{ml}^{(1,2)} = \mathbf{P}_{ml}\, z_l'(\alpha) + \sqrt{l(l+1)}\mathbf{B}_{ml}\, z_l(\alpha)/\alpha \tag{1}$$

$$\mathbf{M}_{ml}^{(1,2)} = \sqrt{l(l+1)}\mathbf{C}_{ml}\, z_l(\beta) \tag{2}$$

$$\mathbf{N}_{ml}^{(1,2)} = l(l+1)\mathbf{P}_{ml}\, z_l(\beta)/\beta + \sqrt{l(l+1)}\mathbf{B}_{ml}\frac{1}{\beta}\frac{d}{d\beta}[\beta z_l(\beta)] \tag{3}$$

$$\alpha = K_\alpha R \qquad \beta = K_\beta R$$

The superscripts (1, 2) refer to the cases $z_l = h_l^{(1)}$ and $z_l = h_l^{(2)}$ respectively. The trio $\mathbf{P}_{ml}$, $\mathbf{C}_{ml}$, $\mathbf{B}_{ml}$ is a set of mutually orthogonal vector spherical harmonics (Morse and Feshbach, 1953) which are constructed from basic solutions of the scalar Helmholtz wave equation

$$\mathbf{P}_{ml} = \mathbf{a}_R\, e^{im\varphi} P_l^m(\cos\theta) \tag{4}$$

$$\sqrt{l(l+1)}\mathbf{C}_{ml} = \mathbf{a}_\theta \frac{im}{\sin\theta} e^{im\varphi} P_l^m(\cos\theta) - \mathbf{a}_\varphi\, e^{im\varphi}\frac{\partial P_l^m(\cos\theta)}{\partial\theta} \tag{5}$$

$$\sqrt{l(l+1)}\mathbf{B}_{ml} = \mathbf{a}_\theta\, e^{im\varphi}\frac{\partial P_l^m(\cos\theta)}{\partial\theta} + \mathbf{a}_\varphi \frac{im}{\sin\theta} e^{im\varphi} P_l^m(\cos\theta) \tag{6}$$

The irrotational solution $\mathbf{L}_{ml}^{(1,2)}$ represents longitudinal motion (P-wave) with a radial component along the vector $\mathbf{P}_{ml}$ and a transverse component aligned with $\mathbf{B}_{ml}$. The solenoidal solution $\mathbf{M}_{ml}$ renders a pure tangential motion and therefore is used to represent SH waves. Likewise, $\mathbf{N}_{ml}$ represents motion of the SV type. Further properties of these vectors are found in the reference cited above.

We next proceed to construct our basic localized directional sources from the vector spherical harmonics. We start with the vertical upward force located at the pole $\theta = 0$ $\varphi = 0$. Clearly, this force is expressible in terms of the vectors $\mathbf{P}_{ml}$ alone with $m = 0$ because of symmetry. Employing the orthogonality relations for the vector spherical harmonics over a sphere with radius $0 < R_{s+1} \leqq a$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} (\mathbf{P}_{0l}\cdot\mathbf{P}_{0l'}^*) \sin\theta\, d\theta = 4\pi/(2l+1) \tag{7}$$

we obtain the representation

$$\frac{\delta(\theta)\delta(\varphi)}{R_{s+1}^2 \sin\theta} \boldsymbol{a_R} = \frac{F_0}{4\pi R_{s+1}^2} \sum_{l=0}^{\infty} (2l+1)\mathbf{P}_{0l} \tag{8}$$

One may verify that the right side of equation (8) is indeed a delta function. Consider, for example, the generating function for the Legendré polynomials,

$$\sum_{l=0}^{\infty} \tau^l P_l(\cos\theta) = (1 - 2\,\tau\cos\theta + \tau^2)^{-1/2} \qquad \tau < 1 \tag{9}$$

A differentiation with respect to τ followed by two arithmetic operations yields

$$f(\tau,\theta) = \sum_{l=0}^{\infty} \tau^l (2l+1) P_l(\cos\theta) = \frac{1-\tau^2}{(1 - 2\tau\cos\theta + \tau^2)^{3/2}} \tag{10}$$

But

$$\lim_{\tau\to 1} f(\tau,\theta) = \begin{Bmatrix} 0 \\ \infty \end{Bmatrix} \qquad \begin{matrix} \text{for } \theta \neq 0 \\ \text{for } \theta = 0 \end{matrix}$$

Indicating that $f(1,\theta)$ is the desired function.

The corresponding representation for an horizontal localized force is obtained in a similar way. We assume a possible expansion in terms of the surface vector harmonics $\mathbf{C}_{ml}$ and $\mathbf{B}_{ml}$

$$\frac{\delta(\theta-\theta_0)\delta(\varphi)}{R_{s+1}^2 \sin\theta} a_\theta = \sum_{l,m=1}^{\infty} \{J_1(l,m)\mathbf{C}_{ml} + J_2(l,m)\mathbf{B}_{ml}\} \tag{11}$$

Using the orthogonality relations,

$$\int_0^{2\pi} d\varphi \int_0^{\pi} (\mathbf{B}_{ml}\cdot\mathbf{B}_{m'l'}^*) \sin\theta\, d\theta = \int_0^{2\pi} d\varphi \int_0^{\pi} (\mathbf{C}_{ml}\cdot\mathbf{C}_{m'l'}^*) \sin\theta\, d\theta$$

$$= \frac{(4\pi/\epsilon_m)}{(2l+1)} \frac{(l-m)!}{(l+m)!} \delta_{mm'}\,\delta_{ll'} \tag{12}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} (\mathbf{B}_{ml}\cdot\mathbf{C}_{m'l'}^*) = 0$$

together with the limits

$$\lim_{\theta\to 0} \frac{P_l^m(\cos\theta)}{\sin\theta} = \lim_{\theta\to 0} \frac{\partial P_l^m(\cos\theta)}{\partial\theta} = \frac{l(l+1)}{2}\,\delta_{ml} \tag{13}$$

we obtain for the unknown coefficients in equation (11)

$$J_1 = -i\,\frac{(2l+1)}{\sqrt{l(l+1)}}\,\frac{1}{4\pi R_{s+1}^2}\,; \qquad J_2 = iJ_1 \tag{14}$$

substituting these values in equation (11) we finally have,

$$\frac{\delta(\theta)\delta(\varphi)}{R_{s+1}^2 \sin\theta}\,\mathbf{a}_\theta = \frac{F_1}{4\pi R_{s+1}^2}\sum_{l=1}^{\infty}\frac{(2l+1)}{\sqrt{l(l+1)}}\,\{\mathbf{B}_{1l} - i\mathbf{C}_{1l}\} \tag{15}$$

where the sum starts at $l = 1$. Although the limits $\mathbf{B}_{1l}/l$, $\mathbf{C}_{1l}/l$ exist for $l = 0$, the sum from zero to infinity will not represent a delta function (see Appendix 1).

Vector Boundary Conditions, Matrix Relations and the Complete Singlet Displacement Field

As stated before, we consider a spherical earth made up of N concentric homogeneous layers.

We assign the label $R_1 = b$ to the core and $R_{N+1} = a$ to the free surface. Let R_j be the inner radius of any layer and R_{s+1} the outer radius of the source layer with the source location at $R = R_{s+1}$, $\theta = 0$, $\varphi = 0$.

We assign to each layer three vector fields $\mathbf{L}_{m,l}^{(1,2)}$, $\mathbf{M}_{m,l}^{(1,2)}$ and $\mathbf{N}_{m,l}^{(1,2)}$, with the exception of the core for which we assign the vectors $\mathbf{L}_{m,l}^{(1)}$ and $\mathbf{N}_{m,l}^{(1)}$.

Consider first the vertical upward-force. Constructing the field in the j-th layer by means of a linear combination of outgoing and incoming radial functions, we expand the total displacement field ${}_j\mathbf{U}_{ml}$ in terms of the vector spherical harmonics (eqs. (1)–(6)).

$$\begin{aligned}{}_j\mathbf{U}_{ml} = \{&{}_jD_{ml}h_l'^{(1)}(\alpha) + {}_jE_{ml}\,h_l'^{(2)}(\alpha) + {}_jF_{ml}\,l(l+1)h_l^{(1)}(\beta)/\beta \\ &+ {}_jG_{ml}\,l(l+1)h_l^{(2)}(\beta)/\beta\}\mathbf{P}_{ml} + \sqrt{l(l+1)}\left\{{}_jD_{ml}\,h_l^{(1)}(\alpha)/\alpha\right. \\ &\left.+ {}_jE_{ml}\,h_l^{(2)}(\alpha)/\alpha + {}_jF_{ml}\frac{d}{d\beta}[\beta h_l^{(1)}(\beta)]/\beta + {}_jG_{ml}\frac{d}{d\beta}[\beta h_l^{(2)}(\beta)]/\beta\right\}\mathbf{B}_{ml}\end{aligned} \tag{16}$$

where the coefficients ${}_jD_m$, ${}_jE_m$, etc. are the strengths of the multipole field to be determined by the boundary and source conditions.

To obtain the components of the radial stress $\mathbf{T}_R$ in terms of the layer coefficients, we use the vector representation ($\bar{\lambda}$ and $\bar{\mu}$ are Lamé constants)

$$\mathbf{T}_R = \bar{\lambda}\mathbf{a}_R \operatorname{div}\mathbf{U} + 2\bar{\mu}\left\{\frac{\partial\mathbf{U}}{\partial R} + \frac{1}{2}(\mathbf{a}_R \times \operatorname{curl}\mathbf{U})\right\} \tag{17}$$

which upon the substitution of ${}_j\mathbf{U}_{ml}$ from equation (16) will render the radial stress vector in terms of the vectors $\mathbf{P}_m$, $\mathbf{B}_m$ and the unknown layer coefficients. Following the convenient procedure of Gilbert and MacDonald (1960) we construct the *spheroidal layer matrix* $\mathfrak{A}_j$ which relates the displacements and radial stresses in the j-th layer to the unknown field strengths of that layer.

$$\begin{bmatrix} U_{RR} \\ U_{\theta R} \\ T_{RR} \\ T_{\theta R} \end{bmatrix} = \begin{bmatrix} \dfrac{d}{d\alpha} h_l^{(1)}(\alpha) & \dfrac{d}{d\alpha} h_l^{(2)}(\alpha) & l(l+1)h_l^{(1)}(\beta)/\beta & l(l+1)h_l^{(2)}(\beta)/\beta \\ h_l^{(1)}(\alpha)/\alpha & h_l^{(2)}(\alpha)/\alpha & \dfrac{1}{\beta}\dfrac{d}{d\beta}[\beta h_l^{(1)}(\beta)] & \dfrac{1}{\beta}\dfrac{d}{d\beta}[\beta h_l^{(2)}(\beta)] \\ \{2h_l''^{(1)}(\alpha) - h_l^{(1)}(\alpha)\}K_\alpha \mu & \{2h_l''^{(2)}(\alpha) - h_l^{(2)}(\alpha)\}K_\alpha \mu & 2l(l+1)\left[\dfrac{h_l^{(1)}(\beta)}{\beta}\right]' K_\beta \mu & 2l(l+1)\left[\dfrac{h_l^{(2)}(\beta)}{\beta}\right]' \\ 2\left[\dfrac{h_l^{(1)}(\alpha)}{\alpha}\right]' K_\alpha \mu & 2\left[\dfrac{h_l^{(2)}(\alpha)}{\alpha}\right]' K_\alpha \mu & \left\{h_l''^{(1)}(\beta) + (l^2+l-2)\dfrac{h_l^{(1)}(\beta)}{\beta}\right\} K_\beta \mu & K_\beta \mu \left\{h_l''^{(2)}(\beta) + (l^2+l-2)\dfrac{h_l^{(2)}(\beta)}{\beta}\right\} \end{bmatrix} \begin{bmatrix} D \\ E \\ F \\ G \end{bmatrix} \tag{18}$$

All physical constants relate to the j-th layer. U_{RR}, $U_{\theta\theta}$ are the radial factors of the total displacement in the directions of the vectors $\mathbf{P}_{ml}$ and $\mathbf{B}_{ml}$ respectively. The coefficient D, E, F, and G have the physical dimension of spectral length ($cm \times$ sec). $\alpha = K_\alpha R$, $\beta = K_\beta R$.

Boundary conditions require continuity of U_{RR}, $U_{\theta R}$, T_{RR} and $T_{\theta R}$ at each of the $(N + 1)$ interfaces, except at the source level where we have from equation (8)

$$(T_{RR})_{s+1} - (T_{RR})_s = L_0/4\pi R_{s+1}^2 \qquad \text{at } R = R_{s+1} \tag{19}$$

One next relates the stress and displacement at the level R_j to the same quantities at level R_{j+1} by means of the layer matrix. Repeating this step for each interface and taking into account the source condition as given in equation (19) one is able to evaluate the unknown layer coefficients. The explicit expressions for the total surficial displacement field becomes

$$U_{RR} = \frac{F_0 e^{i\omega t}}{4\pi\mu_s R_{s+1}^2} \sum_{l=0}^{\infty} (2l + 1) P_l(\cos\theta) N_{RR}(\omega, R_{s+1}, l) \tag{20}$$

$$U_{\theta R} = \frac{F_0 e^{i\omega t}}{4\pi\mu_s R_{s+1}^2} \sum_{l=0}^{\infty} (2l + 1) \frac{\partial P_l(\cos\theta)}{\partial\theta} N_{\theta R}(\omega, R_{s+1}, l) \tag{21}$$

$$U_{\varphi R} = 0 \tag{22}$$

where

$$N_{RR} = \frac{C_{33}^S C_{42}^N - C_{43}^S C_{32}^N}{C_{31}^N C_{42}^N - C_{32}^N C_{41}^N} \tag{23}$$

$$N_{\theta R} = \frac{C_{43}^S C_{31}^N - C_{33}^S C_{41}^N}{C_{31}^N C_{42}^N - C_{32}^N C_{41}^N} \tag{24}$$

and C_{ik}^j are the elements of the matrix $\mathfrak{C}^j$ defined by

$$\mathfrak{C}^j = \mathfrak{B}_1 \mathfrak{B}_2 \cdots \mathfrak{B}_j \tag{25}$$

$$\mathfrak{B}_j = \mathfrak{A}_j(R_j) \mathfrak{A}_j^{-1}(R_{j+1}) \tag{26}$$

$\mathfrak{A}_j$ is the layer matrix in equation (18). Further general properties of these matrices for the case of a stratified half-space were explored by Harkrider (1964).

We turn now to evaluate the response of the medium to an horizontal force, pointing in the direction of the increasing colatitudinal angle θ. The derivation of the layer matrix proceeds similarly to the derivation of the matrix $\mathfrak{A}_j$ for the vertical force. There are here six equations for each interface which split into two independent groups: two equations for the toroidal motion in the coefficients A, B and four equations in the coefficients D, E, F, and G for the spheroidal motion. The *toroidal layer-matrix* is defined by:

$$\begin{bmatrix} U_t \\ T_{\varphi R} \end{bmatrix} = \begin{bmatrix} h_l^{(1)}(\beta) & h_l^{(2)}(\beta) \\ K_\beta \mu\{h_l'^{(1)}(\beta) - h_l^{(1)}(\beta)/\beta\} & K_\beta \mu\{h_l'^{(2)}(\beta) - h_l^{(2)}(\beta)/\beta\} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \tag{27}$$

refers to the radial factor of the toroidal field. This field is given explicitly by

$$U_{\theta\theta} = \frac{F_1\, e^{i\omega t}}{4\pi\mu_s\, R_{s+1}^2} \cos\varphi \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \frac{P_l^1(\cos\theta)}{\sin\theta} N_\theta(\omega, R_{s+1}, l) \tag{28}$$

$$U_{\varphi\theta} = \frac{-F_1\, e^{i\omega t}}{4\pi\mu_s\, R_{s+1}^2} \sin\varphi \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \frac{\partial P_l^1(\cos\theta)}{\partial\theta} N_\varphi(\omega, R_{s+1}, l) \tag{29}$$

where

$$N_\theta \equiv N_\varphi = d_{22}^s/d_{22}^N \tag{30}$$

and d_{ik} is defined with respect to the toroidal matrix analogous to the definition of C_{ik} with respect to the spheroidal matrix. Equation (30) was previously derived by Gilbert and MacDonald (1960).

The spheroidal matrix for the horizontal force is identical with that for the vertical force, since this matrix is a property of the medium and not of the source. The surface displacements for this case become

$$U_{R\theta} = \frac{F_1\, e^{i\omega t}}{4\pi R_{s+1}^2\, \mu_s} \cos\varphi \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} P_l^1(\cos\,\theta) N_{R\theta}(\omega, R_{s+1}, l) \tag{31}$$

$$U_{\theta\theta} = \frac{F_1\, e^{i\omega t}}{4\pi R_{s+1}^2\, \mu_s} \cos\varphi \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \frac{\partial P_l^1(\cos\theta)}{\partial\theta} N_{\theta\theta}(\omega, R_{s+1}, l) \tag{32}$$

$$U_{\varphi\theta} = \frac{F_1\, e^{i\omega t}}{4\pi R_{s+1}^2\, \mu_s} \sin\varphi \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \frac{P_l^1(\cos\theta)}{\sin\theta} N_{\varphi\theta}(\omega, R_{s+1}, l) \tag{33}$$

with

$$N_{R\theta} = \frac{C_{34}^s\, C_{42}^N - C_{44}^s\, C_{32}^N}{C_{31}^N\, C_{42}^N - C_{32}^N\, C_{41}^N} \tag{34}$$

$$N_{\varphi\theta} = N_{\theta\theta} = \frac{C_{44}^s\, C_{42}^N - C_{34}^s\, C_{41}^N}{C_{31}^N\, C_{42}^N - C_{32}^N\, C_{41}^N} \tag{35}$$

The Displacement Dyadic and the Construction of the Dipolar Displacement Field

It was shown in a previous article (Ben-Menahem and Harkrider, 1964) that one may obtain the medium response to dipolar sources from the basic singlet transfer function N_{RR}, $N_{\theta R}$, N_θ, $N_{\theta R}$ and $N_{\theta\theta}$ given earlier in equations (23), (24), (30), (34) and (35). Proceeding along the same lines, we consider a single force $\mathbf{F}$ bound at $R = R_{s+1}$, $\theta = 0$, $\varphi = 0$, with unrestricted orientation. We express this vector in the source coordinate system in the form:

$$\mathbf{F} = |F|\{\sin\lambda\, \sin\delta\mathbf{a}_R + \cos\lambda\mathbf{a}_\theta + \sin\lambda\, \cos\delta\mathbf{a}_\varphi\} \tag{36}$$

If one associates with this motion vector a fault striking in the $\boldsymbol{a}_\theta$ direction, then one will be able to interpret δ as the dip angle (taken positively downward from the free surface) and λ as the slip angle (taken positively counter-clockwise from the positive strike direction in the plane of motion). The convention is followed that all three force components are positive on the hanging-wall side of a reverse left lateral fault.

The displacement field due to $\mathbf{F}$ is constructed from the vertical force response (equations 20–22) with $F_0 = \sin\lambda \sin\delta$ plus the colatitudinal response (equations 28–33) with $F_1 = \cos\lambda$. The contribution from the azimuthal component is obtained by setting the value $\sin\lambda\cos\delta$ for F_1 in equations 28–31, and then rotating these solutions by $(-\pi/2)$ radians in the φ direction. The spectral response of the layered sphere to the excitation $\mathbf{F}(i\omega)$ is given by the totality of these solutions.

It is convenient to define the *singlet displacement dyadic* whose element U_{jk} is the displacement component in the a_j direction ($j = R, \theta,$) due to a *unit force* in the a_k direction ($k = R, \theta,$). Suppressing the summation over l, together with the common factor $e^{i\omega t}/4\pi\mu_s R_s{}^2{}_{+1}$, we shall have for these elements:

First row from left (total vertical displacement)

$$U_{RR} = (2l+1)P_l(\cos\theta)N_{RR} \qquad l = 0, 1, 2, \cdots \tag{37}$$

$$\left.\begin{aligned} U_{R\theta} &= \frac{(2l+1)}{l(l+1)}\cos\varphi P_l^1(\cos\theta)N_{R\theta} \qquad (38)\\ U_{R\varphi} &= \frac{(2l+1)}{l(l+1)}\sin\varphi P_l^1(\cos\theta)N_{R\theta} \qquad (39)\end{aligned}\right\} l = 1, 2, 3, \cdots$$

Second row (total colatitudinal displacement)

$$U_{\theta R} = (2l+1)\frac{\partial P_l(\cos\theta)}{\partial\theta}N_{\theta R} \qquad l = 0, 1, 2, \cdots \tag{40}$$

$$U_{\theta\theta} = \frac{(2l+1)}{l(l+1)}\cos\varphi\left\{\frac{\partial P_l^1(\cos\theta)}{\partial\theta}N_{\theta\theta} + \frac{P_l^1(\cos\theta)}{\sin\theta}N_\theta\right\} \tag{41}$$

$$U_{\theta\varphi} = \frac{(2l+1)}{l(l+1)}\sin\varphi\left\{\frac{\partial P_l^1(\cos\theta)}{\partial\theta}N_{\theta\theta} + \frac{P_l^1(\cos\theta)}{\sin\theta}N_\theta\right\} \tag{42}$$

Third row (total azimuthal displacement)

$$U_{\varphi R} = 0 \tag{43}$$

$$U_{\varphi\theta} = \frac{(2l+1)}{l(l+1)}\sin\varphi\left\{\frac{P_l^1(\cos\theta)}{\sin\theta}N_{\theta\theta} - \frac{\partial P_l^1(\cos\theta)}{\partial\theta}N_\theta\right\} \tag{44}$$

$$U_{\varphi\varphi} = -\frac{(2l+1)}{l(l+1)}\cos\varphi\left\{\frac{P_l^1(\cos\theta)}{\sin\theta}N_{\theta\theta} - \frac{\partial P_l^1(\cos\theta)}{\partial\theta}N_\theta\right\} \tag{45}$$

The total field due to the excitation $\mathbf{F}(i\omega)$ can now be written in the convenient matrix form as the inner product of the displacement dyadic and the force vector,

$$\begin{bmatrix} U_R^s \\ U_\theta^s \\ U_\varphi^s \end{bmatrix} = \begin{bmatrix} U_{RR} & U_{R\theta} & U_{R\varphi} \\ U_{\theta R} & U_{\theta\theta} & U_{\theta\varphi} \\ 0 & U_{\varphi\theta} & U_{\varphi\varphi} \end{bmatrix} \begin{bmatrix} \sin\lambda \sin\delta \\ \cos\lambda \\ \sin\lambda\cos\delta \end{bmatrix} \tag{46}$$

The superscript s indicates that the field is due to a single force.

To obtain the field due to a couple source (dipole with moment) at the origin ($R = R_{s+1}$, $\theta = 0$, $\varphi = 0$) we introduce the vector $\mathbf{n}$ which is orthogonal both to $\mathbf{F}$ and to the plane of motion

$$\mathbf{n} = |n|\{\cos\delta \mathbf{a}_R - \sin\delta \mathbf{a}_\varphi\} \tag{47}$$

The total displacement $\mathbf{U}^c$ at the recording station ($R = a, \theta, \varphi$) is obtained by the application of a differential dyadic operator to the singlet field,

$$\mathbf{U}^c = (\mathbf{n}\cdot\mathrm{grad}_{(\mathrm{source})})\mathbf{U}^s \tag{48}$$

The gradient should be taken in the coordinate system of the source, since the displacement $\mathbf{U}^s$ is expressed in the same system. However, the construction of the multipole source is actually done by differentiating the single force in the station system. Thus we must operate on the physical components of the gradient vector by an orthogonal transformation which corresponds to the two rotations necessary to move the station system back to the source system. It then follows from equations (47) and (48)

$$\mathbf{U}^c = \begin{bmatrix} -\cos\delta \\ 0 \\ \sin\delta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta\cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & \cos\theta\cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial R} \\ \dfrac{\partial}{R\partial\theta} \\ \dfrac{\partial}{R\sin\theta\partial\varphi} \end{bmatrix} \cdot \mathbf{U}^s \tag{49}$$

The dyadic operator to the left of $\mathbf{U}^s$ transforms each component of $\mathbf{U}^s$ into the corresponding component of $\mathbf{U}^c$.

The motion due to a *double couple* is obtained by the construction of an additional couple in which the roles of the force $\mathbf{F}$ and the normal $\mathbf{n}$ have been interchanged and the superposition of this field on the original couple field. The first stage is fulfilled merely by interchanging the force-components column in equation (46) by the normal components column in equation (49).

The final results can be given in a form which emphasizes the dependence on the azimuth angle φ,

$$\begin{aligned} U_R^{DC}/(2l+1) &= (a_0 \sin\lambda\sin 2\delta) + (a_1 \sin\lambda\cos 2\delta)\sin\varphi \\ &+ (a_2\cos\lambda\cos\delta)\cos\varphi + (a_3\cos\lambda\sin\delta)\sin 2\varphi + (a_4\sin\lambda\sin 2\delta)\cos 2\varphi \end{aligned} \tag{50}$$

where

$$
\begin{aligned}
a_0 = {} & \frac{1}{2l(l+1)} \frac{P_l^1(\cos\theta)}{\sin\theta} \frac{N_{R\theta}}{R_{s+1}} + \sin\theta P_l^1(\cos\theta) \left[\frac{1}{2l(l+1)} \frac{\partial N_{R\theta}}{\partial R_{s+1}} - \frac{N_{RR}}{R_{s+1}} \right] \\
& + \cos\theta \left[\frac{1}{2l(l+1)} \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \frac{N_{R\theta}}{R_{s+1}} - P_l(\cos\theta) \frac{\partial N_{RR}}{\partial R_{s+1}} \right]
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
a_1 = a_2 = {} & -\cos\theta P_l^1(\cos\theta) \left[\frac{1}{l(l+1)} \frac{\partial N_{R\theta}}{\partial R_{s+1}} - \frac{N_{RR}}{R_{s+1}} \right] \\
& + \sin\theta \left[\frac{\frac{\partial}{\partial\theta} P_l^1(\cos\theta)}{l(l+1)} \frac{N_{R\theta}}{R_{s+1}} - P_l(\cos\theta) \frac{\partial N_{RR}}{\partial R_{s+1}} \right]
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
a_3 = -2a_4 = {} & \sin\theta \frac{P_l^1(\cos\theta)}{l(l+1)} \frac{\partial N_{R\theta}}{\partial R_{s+1}} \\
& + \cos\theta \frac{\frac{\partial P_l^1(\cos\theta)}{\partial\theta}}{l(l+1)} \frac{N_{R\theta}}{R_{s+1}} - \frac{1}{l(l+1)} \frac{P_l^1(\cos\theta)}{\sin\theta} \frac{N_{R\theta}}{R_{s+1}}
\end{aligned} \tag{53}
$$

Each term in the coefficients a_i passes to a finite limit as $R_0 \rightarrow \infty$. It can be readily checked with the aid of Appendix 3 that these limits reduce to the corresponding coefficients given by Ben-Menahem and Harkrider (1964) for the stratified half-space.

Likewise,

$$
\begin{aligned}
U_\varphi^{DC}/(2l+1) = {} & (b_1 \cos\lambda \cos\delta) \sin\varphi + (b_2 \sin\lambda \cos 2\delta) \cos\varphi \\
& + (b_3 \sin\lambda \sin 2\delta) \sin 2\varphi + (b_4 \cos\lambda \sin\delta) \cos 2\varphi
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
b_1 = -b_2 = {} & \frac{\sin\theta}{l(l+1)} \left[\frac{N_{\theta\theta}}{R_{s+1}} \frac{\partial}{\partial\theta} \left\{ \frac{P_l^1(\cos\theta)}{\sin\theta} \right\} \right. \\
& \left. - \frac{N_\theta}{R_{s+1}} \frac{\partial^2 P_l^1(\cos\theta)}{\partial\theta^2} \right] \\
& - \frac{\cos\theta}{l(l+1)} \left[\frac{P_l^1(\cos\theta)}{\sin\theta} \frac{\partial N_{\theta\theta}}{\partial R_{s+1}} - \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \frac{\partial N_\theta}{\partial R_s} \right]
\end{aligned} \tag{55}
$$

$$
\begin{aligned}
2b_3 = b_4 = {} & \frac{1}{l(l+1)\sin\theta R_{s+1}} \left[\frac{P_l^1(\cos\theta)}{\sin\theta} N_{\theta\theta} - \frac{\partial P_l^1}{\partial\theta} N_\theta \right] \\
& - \frac{\sin\theta}{l(l+1)} \left[\frac{P_l^1(\cos\theta)}{\sin\theta} \frac{\partial N_{\theta\theta}}{\partial R_{s+1}} - \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \frac{\partial N_\theta}{\partial R_s} \right] \\
& - \frac{\cos\theta}{l(l+1)} \left[\frac{N_{\theta\theta}}{R_s} \frac{\partial}{\partial\theta} \left\{ \frac{P_l^1(\cos\theta)}{\sin\theta} \right\} - \frac{N_\theta}{R_s} \frac{\partial^2 P_l^1(\cos\theta)}{\partial\theta^2} \right]
\end{aligned} \tag{56}
$$

and

$$U_\theta^{DC}/(2l+1) = (c_0 \sin\lambda \sin 2\delta) + (c_1 \sin\lambda \cos 2\delta)\sin\varphi + (c_2\cos\lambda\cos\delta)\cos\varphi + (c_3\cos\lambda\sin\delta)\sin 2\varphi + (c_4\sin\lambda\sin 2\delta)\cos 2\varphi \tag{57}$$

where

$$\begin{aligned} c_0 = {} & \frac{1}{2l(l+1)}\frac{1}{R_{s+1}\sin\theta}\left[\frac{\partial P_l^1(\cos\theta)}{\partial\theta}N_{\theta\theta} + \frac{P_l^1(\cos\theta)}{\sin\theta}\right] \\ & + \sin\theta\left[\frac{\partial^2 P_l(\cos\theta)}{\partial\theta^2}\frac{N_{\theta R}}{R_{s+1}} + \frac{1}{2l(l+1)}\left\{\frac{\partial P_l^1(\cos\theta)}{\partial\theta}\frac{\partial N_{\theta\theta}}{\partial R_{s+1}}\right.\right. \\ & \left.\left. + \frac{P_l^1(\cos\theta)}{\sin\theta}\frac{\partial N_\theta}{\partial R_{s+1}}\right\}\right] - \cos\theta\left[\frac{\partial P_l(\cos\theta)}{\partial\theta}\frac{\partial N_{\theta R}}{\partial R_{s+1}}\right. \\ & \left. - \frac{1}{2l(l+1)}\left\{\frac{\partial^2 P_l^1(\cos\theta)}{\partial\theta^2}\frac{N_{\theta\theta}}{R_{s+1}} + \frac{\partial}{\partial\theta}\frac{P_l^1(\cos\theta)}{\sin\theta}\frac{N_\theta}{R_{s+1}}\right\}\right] \end{aligned} \tag{58}$$

$$\begin{aligned} c_1 = c_2 = {} & \sin\theta\left[\frac{1}{l(l+1)}\left\{\frac{\partial^2 P_l^1(\cos\theta)}{\partial\theta^2}\frac{N_{\theta\theta}}{R_{s+1}} + \frac{\partial}{\partial\theta}\frac{P_l^1(\cos\theta)}{\sin\theta}\frac{N_\theta}{R_{s+1}}\right\}\right. \\ & \left. - \frac{\partial P_l(\cos\theta)}{\partial\theta}\frac{\partial N_{\theta R}}{\partial R_{s+1}}\right] - \cos\theta\left[\frac{1}{l(l+1)}\left\{\frac{\partial P_l^1(\cos\theta)}{\partial\theta}\frac{\partial N_{\theta\theta}}{\partial R_{s+1}}\right.\right. \\ & \left.\left. + \frac{P_l^1(\cos\theta)}{\sin\theta}\frac{\partial N_\theta}{\partial R_{s+1}}\right\} + \frac{\partial^2 P_l(\cos\theta)}{\partial\theta^2}\frac{N_{\theta R}}{R_{s+1}}\right] \end{aligned} \tag{59}$$

$$\begin{aligned} c_3 = -2c_4 = {} & \frac{\sin\theta}{l(l+1)}\left[\frac{\partial P_l^1(\cos\theta)}{\partial\theta}\frac{\partial N_{\theta\theta}}{\partial R_{s+1}} + \frac{P_l^1(\cos\theta)}{\sin\theta}\frac{\partial N_\theta}{\partial R_{s+1}}\right] \\ & + \frac{\cos\theta}{l(l+1)}\left[\frac{\partial^2 P_l^1(\cos\theta)}{\partial\theta^2}\frac{N_{\theta\theta}}{R_{s+1}} + \frac{N_{\theta\theta}}{R_{s+1}}\frac{\partial}{\partial\theta}\frac{P_l^1(\cos\theta)}{\sin\theta} + \frac{\partial}{\partial\theta}\frac{P_l^1(\cos\theta)}{\sin\theta}\frac{N_\theta}{R_{s+1}}\right] \\ & - \frac{1}{l(l+1)R_s\sin\theta}\left[\frac{\partial P_l^1(\cos\theta)}{\partial\theta}N_{\theta\theta} + \frac{P_l^1(\cos\theta)}{\sin\theta}N_\theta\right] \end{aligned} \tag{60}$$

If one wishes, one may further simplify the coefficients a_i, b_i and c_i by using the relation (1–2). The present form, however, lends itself easier to approximations.

The foregoing equations (50)–(60) express the total field in and on the sphere due to a double-couple source at depth $R = R_{s+1}$. The dependence of the displacements on the azimuth angle φ, generally known as the *'radiation pattern'* is somewhat similar to the parallel case of the half-space. However, one of the striking differences is the non-vanishing of the derivatives with respect to the depth for a vertical fault ($\delta = 90°$) because of the earth's curvature.

Note also that both the radial and the colatitudinal displacements have terms which are independent of the azimuth angle.

The mode ${}_1S_0(l = 0)$ known as the 'radial mode' produces the simplest field,

$$
\begin{aligned}
U^S &= \sin\lambda \sin\delta N_{RR} \\
U^C &= \sin\lambda \sin\delta(\sin\delta \sin\theta \sin\varphi - \cos\delta \cos\theta) \frac{\partial N_{RR}}{\partial R_{s+1}} \\
U^{DC} &= -\{\cos\lambda \cos\delta \sin\theta \cos\varphi \\
&\qquad + \sin\lambda(\sin 2\delta \cos\theta + \cos 2\delta \sin\theta \sin\varphi)\} \frac{\partial N_{RR}}{\partial R_{s+1}}
\end{aligned}
\tag{61}
$$

The complexity of the displacement field starts at once already for $l = 1$. To get further information about the relative amplitudes of the components, numerical calculations must be applied.

Long-Period Surface Waves

The formal results which were obtained in the previous paragraphs can be used to evaluate the amplitude response functions for surface waves with periods up to 450 sec ($l \geqq 14$, $\lambda_0 \leqq 3000$ km). For longer periods (lower values of l) one must consider the effect of the gravitational forces and the presence of the earth's core. From a practical point of view this threshold is also dictated by our ability to observe these long periods with sufficient amount of power in individual spectrums of surface waves. We found from the analysis of records of some major earthquakes that waves longer than 450 sec do not have sufficient power in G_1–G_8 and R_1–R_5. Arrivals of higher order interfere with each other and therefore cannot be considered as transients any longer.

Ben-Menahem and Harkrider (1964) computed the amplitude and phase response of a layered flat earth to a class of spatial excitation functions. Their scheme can be reformulated for a spherical earth and their results can thus be extended to the above period threshold.

To separate the surface waves from the complete displacement field, we apply Watson's transformation to each element of the 'field dyadic'. Denoting the surface wave contribution of each element by a bar superscript, we have for example

$$
\bar{U}_{RR} = -\pi \sum_{l_q} \frac{N(l_q, \omega, R_{s+1})}{\sin \pi l_q \left\{ \dfrac{\partial F(l, \omega)}{\partial l} \right\}_{l=l_q}} P_{l_q}[\cos(\pi - \theta)] \tag{62}
$$

where l_q is the q-th root of the transcendental period equation $F = C_{31}^N C_{42}^N - C_{32}^N C_{41}^N = 0$ with l as a continuous variable and $N = C_{33}^S C_{42}^N - C_{43}^S C_{32}^N$ (see equation (23)).

Each root represents a different mode and the sum extends over the infinite mode set.

Next we use the well-known asymptotic expansion of $P_l^m(\cos\theta)$ and decompose the standing oscillation in equation (64) into two waves travelling in opposite directions,

$$P_l^m(\cos\theta) \cong \sqrt{\frac{l^{2m}}{2\pi l \sin\theta}} \left\{ \exp i\left[\left(l+\frac{1}{2}\right)\theta - m\frac{\pi}{2} - \frac{\pi}{4}\right] + \exp -i\left[\left(l+\frac{1}{2}\right)\theta - \frac{m\pi}{2} - \frac{\pi}{4}\right]\right\} \tag{63}$$

$$l \gg m \qquad \epsilon < \theta < \pi - \epsilon \qquad l \gg 1$$

This approximation introduces an error of about 2.0 per cent at $T = 450$ sec and 1.0 per cent at $T = 325$ sec. If an additional term of the expansion is taken, the error is decreased to 0.14 per cent at $T = 450$ sec and .05 per cent at $T = 325$ sec. The role of the factor $(\sin \pi l)^{-1} = -2 \sum_{P=0}^{\infty} (-)^P \exp[-\pi i(l + \frac{1}{2})\ (1 + 2P)]$ constitutes in generating the multiple arrivals around the sphere. If we consider only the first arrival of the q-th mode along the minor arc, we may rewrite equation (62) in the form

$$\bar{U}_{RR} = -\pi i \frac{N(l_q, \omega, R_{s+1})}{\left\{\dfrac{\partial F(l, \omega)}{\partial l}\right\}_{l=l_q}} \sqrt{\frac{2}{\pi l_q \sin\theta}} \exp -i\left[\left(l_q + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] \tag{64}$$

The numerical evaluation of the amplitude factor for the dipolar response functions requires the following operations:

1) Solving for the roots l_q of the period equation $F(l, \omega) = 0$. This has been carried out successfully in an earth-stretching approximation (Anderson and Toksoz, 1963) up to period of 475 sec ($l \geqq 14$).
2) Computation of $N(l_q, \omega, R_{s+1})$ and its first derivative with depth $\partial N/\partial R_{s+1}$. Although this was done by Harkrider (1964) for a flat layering only, his program can be fitted to a spherical layering with no considerable changes.
3) The evaluation of the derivative $\partial F(l, \omega)/\partial l$ does not pose serious difficulties. Here one has two alternatives: We may use the identity $dF = 0$, and thence

$$\frac{\partial F}{\partial l} = -\frac{U_g}{a}\frac{\partial F}{\partial \omega} \tag{65}$$

where $U_g = a(\partial\omega/\partial l)$ is the group velocity. Since F incorporates already derivatives with respect to $K_{\alpha_i}R_i$ and $K_{\beta_i}R_i$, no new programming subroutine will be called for.

The second choice is to write $(\partial j_l/\partial l)_{l=l_q} = (\partial j_{l_q}/\partial l_q)_{l_q=l}$ and use the known relations (Oberhettinger, 1958)

$$\left.\frac{\partial j_{l_q}(z)}{\partial l_q}\right|_{l_q=0} = j_0(z)C_i(2z) + n_0(z)S_i(2z) \tag{66}$$

$$\left.\frac{\partial n_{l_q}(z)}{\partial l_q}\right|_{l_q=0} = j_0(z)[S_i(2z) - \pi] - n_0(z)C_i(2z) \tag{67}$$

The recurrence relations for the spherical Bessel functions enable us to generalize these relations for an arbitrary integer $l_q = l$.

It remains to obtain the coefficients a_i, b_i and c_i for this case. If we exclude the polar regions $0 \leqq \theta < \theta_0$ we may assume that $l \sin\theta \gg 1$ and $R_{s+1} \sin\theta \gg 1$. Under these conditions, the coefficients undergo a considerable simplification. Suppressing a common factor $\cos\theta P_l(\cos\theta)$ we find,

$$a_0 = \left[\frac{N_{R\theta}}{2R_{s+1}} - \frac{\partial N_{RR}}{\partial R_{s+1}}\right] + i\epsilon\left[\frac{R_s}{2l^2}\frac{\partial N_{R\theta}}{\partial R_{s+1}} - N_{RR}\right] \tag{68}$$

$$a_1 = a_2 = i\left[\frac{l}{R_{s+1}} N_{RR} - \frac{1}{l}\frac{\partial N_{R\theta}}{\partial R_{s+1}}\right] + \epsilon\left[\frac{N_{R\theta}}{2l} - \frac{R_{s+1}}{l}\frac{\partial N_{RR}}{\partial R_{s+1}}\right] \tag{69}$$

$$a_3 = -2a_4 = \frac{N_{R\theta}}{R_{s+1}} + i\epsilon\frac{R_{s+1}}{l^2}\frac{\partial N_{R\theta}}{\partial R_{s+1}} \tag{70}$$

$$b_1 = -b_2 = \frac{\partial N_\theta}{\partial R_{s+1}} - i\epsilon N_\theta \tag{71}$$

$$b_4 = 2b_3 = i\frac{l}{R_{s+1}} N_\theta + \epsilon\frac{R_{s+1}}{l}\frac{\partial N_\theta}{\partial R_{s+1}} \tag{72}$$

$$c_0 = il\left[\frac{\partial N_{\theta R}}{\partial R_{s+1}} - \frac{1}{2}\frac{N_{\theta\theta}}{R_{s+1}}\right] + \epsilon\left[\frac{R_{s+1}}{2l}\frac{\partial N_{\theta\theta}}{\partial R_{s+1}} - lN_{\theta R}\right] \tag{73}$$

$$c_1 = \left[l^2\frac{N_{\theta R}}{R_{s+1}} - \frac{\partial N_{\theta\theta}}{\partial R_{s+1}}\right] + i\epsilon\left[R_{s+1}\frac{\partial N_{\theta R}}{\partial R_{s+1}} - N_{\theta\theta}\right] \tag{74}$$

$$c_3 = -i\frac{l}{R_{s+1}} N_{\theta\theta} + \epsilon\frac{R_{s+1}}{l}\frac{\partial N_{\theta\theta}}{\partial R_{s+1}} \tag{75}$$

The factor $\epsilon = (l/R_{s+1})tg\theta$ is due to the curvature of the sphere. If one compares the coefficients in equations (68–75) with the corresponding coefficients for the case of the half-space (Ben-Menahem and Harkrider, 1964) one will find that the two sets of expressions become identical for $\epsilon = 0$. Hence we identify the terms which include ϵ as the *curvature correction.* The functions N_{RR}, $N_{R\theta}$, $N_{\theta\theta}$, N_θ are transformed into the corresponding half-space functions N_{zz}, N_{zr}, N_{rz}, N_{rr}, and N_θ. Note that the curvature introduces additional phase shifts.

Appendix 1

Equation (15) can also be written as

$$\cos\varphi \sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left\{ \frac{P_l^1(\cos\theta)}{\sin\theta} + \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \right\} = 4\pi \frac{\delta(\theta)\delta(\varphi)}{\sin\theta} \tag{1-1}$$

We wish to prove that the infinite sum is indeed a delta function. We make use of some known relations between the Legendré functions (Morse and Feshbach, 1953)

$$\frac{\partial P_l^1(\cos\theta)}{\partial\theta} + \cos\theta \frac{P_l^1(\cos\theta)}{\sin\theta} = l(l+1)\, P_l(\cos\theta) \tag{1-2}$$

$$\frac{(2l+1)}{l(l+1)} P_l^1(\cos\theta) = \frac{P_{l-1}(\cos\theta) - P_{l+1}(\cos\theta)}{\sin\theta} \tag{1-3}$$

Consequently

$$\begin{aligned} \frac{(2l+1)}{l(l+1)} \left[\frac{P_l^1(\cos\theta)}{\sin\theta} + \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \right] &= (2l+1) P_l(\cos\theta) \\ &\quad + \frac{1-\cos\theta}{\sin^2\theta} \Big[P_{l-1}(\cos\theta) - P_{l+1}(\cos\theta) \Big] \end{aligned} \tag{1-4}$$

summing over l we find

$$\begin{aligned} &\sum_{l=1}^{\infty} \frac{(2l+1)}{l(l+1)} \left[\frac{P_l^1(\cos\theta)}{\sin\theta} + \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \right] \\ &\qquad = \sum_{l=1}^{\infty} (2l+1)\, P_l(\cos\theta) + 1 = \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta) \end{aligned} \tag{1-5}$$

which concludes our proof.

Appendix 2

Seismographs are being described in many articles. Although the mechanism of the various instruments is well exposed, little is said about the sensitivity of these sensors to different portions of the total field. We shall give here just a few examples.

Vertical seismographs of any kind record only spheroidal modes (free oscillations, Rayleigh waves, P and SV type signals) and thus constitute the simplest device of removing the toroidal field.

The elimination of the spheroidal modes from the total field is done by sensors which are capable of detecting the radial (Z component in cartesian coordinates) component of the curl of the displacement. Recalling that $\mathbf{U} = A_1\mathbf{L} + A_2\mathbf{M} + A_3\mathbf{N}$ we find $\mathbf{\Omega} = \text{curl}\ \mathbf{U} = B_2\mathbf{M} + B_3\ \text{curl}\ \mathbf{M}$. But since $\mathbf{M}_R = 0$, we have

$$\mathbf{\Omega}_R \propto (\text{curl}\ \mathbf{M})_R \tag{2-1}$$

In cartesian coordinates $\Omega_z = (\partial u_y/\partial x) - (\partial u_x/\partial y)$. This second order entity is recorded on special rotational strain seismometers and allied devices.

The combination of the output of two linear strain seismometers at right angles is another well-known device which removes the toroidal field. The extension of a line element in the direction (p, q) is given by $p^2 e_{xx} + q^2 e_{yy} + pq e_{xy}$. The extension in the perpendicular direction $(q, -p)$ is therefore $q^2 e_{xx} + p^2 e_{yy} - pq e_{xy}$. Their sum is simply $e_{xx} + e_{xy}$. This operation is equivalent to taking the divergence of the field and since div $\mathbf{M} = 0$ the toroidal modes are eliminated. The enhanced field is div $\mathbf{L} = \nabla^2\psi = K_\alpha^2\psi$, where ψ is the longitudinal potential. This result also shows that $e_{xx} + e_{xy}$ is independent of the directionality of the waves.

Finally there exists the device of the virtual rotation of the horizontal instrumental components (e.g. NS, EW) in order to separate the radial and transversal field components (cylindrical system). This requires a pre-knowledge of the waves directionality. The method is useful only for the far field of body and surface waves provided the station is not located in a nodal line. It cannot be used to separate spheroidal and toroidal parts of the non-radial field. It is of some interest to note that two perpendicular linear strain seismographs can be used to measure the rotational strain due to an incoming plane wave with preknown directionality.

Appendix 3

Consider a source at a point $R = R_{s+1}$, $\theta = \varphi = 0$ and a station at $R = a$. If $a \to \infty$, $R_{s+1} \to \infty$ in such a way that $a - R_{s+1} = h$ remains unchanged, the sphere is transformed into an half-space. Coordinates, variables, and eigenfunctions of the spherical system go into the corresponding members in the cylindrical system (kr, θ, z).

Coordinates, Derivatives, Wave-numbers

$$a \to \infty \qquad \theta \to 0 \qquad \lim a\theta \to r \qquad a - R \to z \qquad \sin\theta \to 0 \qquad \cos\theta \to 1 \tag{3-1}$$

$$l \to \infty \qquad l\theta \to Kr \qquad \frac{l}{a} \to K \qquad \frac{1}{l} \to \frac{dK}{K} \qquad \frac{1}{a} \to dK$$
$$\frac{1}{a}\frac{\partial}{\partial\theta} \to \frac{\partial}{\partial r} \qquad \frac{1}{l}\frac{\partial}{\partial\theta} \to \frac{\partial}{\partial Kr} \tag{3-2}$$

$$\frac{1}{a^2}\sum_0^\infty l\,\{\cdots\} \to \int_0^\infty K\{\cdots\}\,dK \tag{3-3}$$

These and the following transformations are meaningful only when infinite sums in the spherical system are transformed into integrals in the corresponding cylindrical system.

Eigenfunctions

The known limit (Magnus and Oberhettinger, 1954)

$$\lim_{l\to\infty} P_l\left(\cos\frac{Kr}{l}\right) = J_0(Kr) \tag{3-4}$$

can be used to show that

$$\frac{1}{l+1} P_l^1(\cos\theta) \to J_1(Kr) \tag{3-5}$$

$$\frac{1}{l(l+1)} \frac{P_l^1(\cos\theta)}{\sin\theta} \to \frac{J_1(Kr)}{Kr} \tag{3-6}$$

$$\frac{1}{l(l+1)} \frac{\partial P_l^1(\cos\theta)}{\partial\theta} \to J_0(Kr) - \frac{J_1(Kr)}{Kr} \tag{3-7}$$

The radial functions are transformed by means of Debye asymptotic series $(l \to \infty, a \to \infty, a - R_{s+1} \to h)$

$$\frac{j_l'(K_\alpha a)}{j_l(K_\alpha a)} \to \frac{\sqrt{K^2 - K_\alpha^2}}{K_\alpha} \qquad \frac{h_l'(K_\alpha a)}{h_l(K_\alpha a)} \to -\frac{\sqrt{K^2 - K_\alpha^2}}{K_\alpha} \tag{3-8}$$

$$\frac{j_l''(K_\alpha a)}{j_l(K_\alpha a)} \to \frac{K^2 - K_\alpha^2}{K_\alpha^2} \qquad a^2 j_l(K_\alpha a)\, h_l(K_\alpha a) \to \frac{1}{2K_\alpha\sqrt{K^2 - K_\alpha^2}} \tag{3-9}$$

$$\frac{j_l(K_\alpha R_{s+1})}{j_l(K_\alpha a)} \to e^{-h\sqrt{k^2 - k_\alpha^2}} \tag{3-10}$$

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References

Anderson, D. L., and M. N. Toksoz
1963. Surface waves on a spherical earth 1. upper mantel structure from Love waves, *J. Geophys. Res.*, 68, 3483–3500.

Ben-Menahem, A., and D. G. Harkrider
1964. Radiation patterns of seismic surface waves from buried dipolar point sources in a flat stratified earth, *J. Geophys. Res.*, 69, 2605–2620.

Gilbert, F., and G. J. F. MacDonald
1960. Free oscillations of the earth: 1. toroidal oscillations, *J. Geophys. Res.*, 65, 675–693.

Harkrider, D. G.
1964. Surface waves in multilayered elastic media: 1. Rayleigh and Love waves from buried sources in a multilayered elastic half-space, *Bull. Seism. Soc. Amer.*, 54, 627–679.

Haskell, N. A.
1953. The dispersion of surface waves on multilayered media, *Bull. Seism. Soc. Amer.*, 43, 17–34.

Jobert, N.
1955. Effet de la courbure de la terre sur les ondes de Love, *Ann. de Geophys.*, 11, 1–48.

Morse, P. M., and H. Feshbach
1953. Methods of theoretical physics, Parts I–II, McGraw-Hill Book Co., New York.

Oberhettinger, F.
1958. On the derivative of Bessel functions with respect to the order, *J. Math. and Phys.*, 37, 75–78.

Pekeris, C. L., and H. Jarosch
1958. The free oscillations of the earth, Contributions in Geophysics, Pergamon Press, Los Angeles.

Yanovskaya, T. B.
1958. The dispersion of Rayleigh waves in a spherical layer, *Izv. Akad. Nauk.* USSR, Geophy. Series, 461–467.

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